TOPOLOGY WITHOUT TEARS[1]

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Version of June 22, 2001[2]

[2] This book is being progressively updated and expanded; it is anticipated that there will be about fifteen chapters in all. Only those chapters which appear in colour have been updated so far. If you discover any errors or you have suggested improvements please e-mail: Sid.Morris@unisa.edu.au

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Introduction

Topology is an important and interesting area of mathematics, the study of which will not only introduce you to new concepts and theorems but also put into context old ones like continuous functions. However, to say just this is to understate the significance of topology. It is so fundamental that its influence is evident in almost every other branch of mathematics. This makes the study of topology relevant to all who aspire to be mathematicians whether their first love is (or will be) algebra, analysis, category theory, chaos, continuum mechanics, dynamics, geometry, industrial mathematics, mathematical biology, mathematical economics, mathematical finance, mathematical modelling, mathematical physics, mathematics of communication, number theory, numerical mathematics, operations research or statistics. Topological notions like compactness, connectedness and denseness are as basic to mathematicians of today as sets and functions were to those of last century.

Topology has several different branches — general topology (also known as point-set topology), algebraic topology, differential topology and topological algebra — the first, general topology, being the door to the study of the others. We aim in this book to provide a thorough grounding in general topology. Anyone who conscientiously studies about the first ten chapters and solves at least half of the exercises will certainly have such a grounding.

For the reader who has not previously studied an axiomatic branch of mathematics such as abstract algebra, learning to write proofs will be a hurdle. To assist you to learn how to write proofs, quite often in the early chapters, we include an aside which does not form part of the proof but outlines the thought process which led to the proof. Asides are indicated in the following manner:

> In order to arrive at the proof, we went through this thought process, which might well be called the "discovery" or "experiment phase".
>
> However, the reader will learn that while discovery or experimentation is often essential, nothing can replace a formal proof.

There are many exercises in this book. Only by working through a good number of exercises will you master this course. Very often we include new concepts in the exercises; the concepts which we consider most important will generally be introduced again in the text.

Harder exercises are indicated by an *.

Acknowledgment. Portions of earlier versions of this book were used at La Trobe University, University of New England, University of Wollongong, University of Queensland, University of South Australia and City College of New York over the last 25 years. I wish to thank those students who criticized the earlier versions and identified errors. Special thanks go to Deborah King for pointing out numerous errors and weaknesses in the presentation. Thanks also go to several other colleagues including Carolyn McPhail, Ralph Kopperman, Rodney Nillsen, Peter Pleasants, Geoffrey Prince and Bevan Thompson who read earlier versions and offered suggestions for improvements. Thanks also go to Jack Gray whose excellent University of New South Wales Lecture Notes "Set Theory and Transfinite Arithmetic", written in the 1970s, influenced our Appendix on Infinite Set Theory.

Chapter 1

Topological Spaces

Introduction

Tennis, football, baseball and hockey may all be exciting games but to play them you must first learn (some of) the rules of the game. Mathematics is no different. So we begin with the rules for topology.

This chapter opens with the definition of a topology and is then devoted to some simple examples: finite topological spaces, discrete spaces, indiscrete spaces, and spaces with the finite-closed topology.

Topology, like other branches of pure mathematics such as group theory, is an axiomatic subject. We start with a set of axioms and we use these axioms to prove propositions and theorems. It is extremely important to develop your skill at writing proofs.

Why are proofs so important? Suppose our task were to construct a building. We would start with the foundations. In our case these are the axioms or definitions – everything else is built upon them. Each theorem or proposition represents a new level of knowledge and must be firmly anchored to the previous level. We attach the new level to the previous one using a proof. So the theorems and propositions are the new heights of knowledge we achieve, while the proofs are essential as they are the mortar which attaches them to the level below. Without proofs the structure would collapse.

So what is a mathematical proof? A mathematical proof is a watertight argument which begins with information you are given, proceeds by logical argument, and ends with what you are asked to prove.

You should begin a proof by writing down the information you are given and then state what you are asked to prove. If the information you are given or what you are required to prove contains technical terms, then you should write down the definitions of those technical terms.

Every proof should consist of complete sentences. Each of these sentences should be a consequence of (i) what has been stated previously or (ii) a theorem, proposition or lemma that has already been proved.

In this book you will see many proofs, but note that mathematics is not a spectator sport. It is a game for participants. The only way to learn to write proofs is to try to write them yourself.

1.1 Topology

1.1.1 Definitions. Let X be a non-empty set. A collection $\mathcal{T}$ of subsets of X is said to be a topology on X if

(i) X and the empty set, $\emptyset$, belong to $\mathcal{T}$,

(ii) the union of any (finite or infinite) number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$, and

(iii) the intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space.

1.1.2 Example. Let $X = \{a, b, c, d, e, f\}$ and

$$\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}.$$

Then $\mathcal{T}_1$ is a topology on X as it satisfies conditions (i), (ii) and (iii) of Definitions 1.1.1. □

1.1.3 Example. Let $X = \{a, b, c, d, e\}$ and

$$\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, e\}, \{b, c, d\}\}.$$

Then $\mathcal{T}_2$ is not a topology on X as the union

$$\{c, d\} \cup \{a, c, e\} = \{a, c, d, e\}$$

of two members of $\mathcal{T}_2$ does not belong to $\mathcal{T}_2$; that is, $\mathcal{T}_2$ does not satisfy condition (ii) of Definitions 1.1.1. □

1.1.4 Example. Let $X = \{a, b, c, d, e, f\}$ and

$$\mathcal{T}_3 = \{X, \emptyset, \{a\}, \{f\}, \{a, f\}, \{a, c, f\}, \{b, c, d, e, f\}\}.$$

Then $\mathcal{T}_3$ is not a topology on X since the intersection

$$\{a, c, f\} \cap \{b, c, d, e, f\} = \{c, f\}$$

of two sets in $\mathcal{T}_3$ does not belong to $\mathcal{T}_3$; that is, $\mathcal{T}_3$ does not have property (iii) of Definitions 1.1.1. □

1.1.5 Example. Let $\mathbb{N}$ be the set of all natural numbers (that is, the set of all positive integers) and let $\mathcal{T}_4$ consist of $\mathbb{N}$, $\emptyset$, and all finite subsets of $\mathbb{N}$. Then $\mathcal{T}_4$ is not a topology on $\mathbb{N}$, since the infinite union

$$\{2\} \cup \{3\} \cup \cdots \cup \{n\} \cup \cdots = \{2, 3, \ldots, n, \ldots\}$$

of members of $\mathcal{T}_4$ does not belong to $\mathcal{T}_4$; that is, $\mathcal{T}_4$ does not have property (ii) of Definitions 1.1.1. □

1.1.6 Definitions. Let X be any non-empty set and let $\mathcal{T}$ be the collection of all subsets of X. Then $\mathcal{T}$ is called the discrete topology on the set X. The topological space $(X, \mathcal{T})$ is called a discrete space.

We note that $\mathcal{T}$ in Definitions 1.1.6 does satisfy the conditions of Definitions 1.1.1 and so is indeed a topology.

Observe that the set X in Definitions 1.1.6 can be any non-empty set. So there is an infinite number of discrete spaces – one for each set X.

1.1.7 Definitions. Let X be any non-empty set and $\mathcal{T} = \{X, \emptyset\}$. Then $\mathcal{T}$ is called the indiscrete topology and $(X, \mathcal{T})$ is said to be an indiscrete space.

Once again we have to check that $\mathcal{T}$ satisfies the conditions of Definitions 1.1.1 and so is indeed a topology.

We observe again that the set X in Definitions 1.1.7 can be any non-empty set. So there is an infinite number of indiscrete spaces – one for each set X.

In the introduction to this chapter we discussed the importance of proofs and what is involved in writing them. Our first experience with proofs is in Example 1.1.8 and Proposition 1.1.9. You should study these proofs carefully.

1.1.8 Example. If $X = \{a, b, c\}$ and $\mathcal{T}$ is a topology on X with $\{a\} \in \mathcal{T}$, $\{b\} \in \mathcal{T}$, and $\{c\} \in \mathcal{T}$, prove that $\mathcal{T}$ is the discrete topology.

Proof.

We are given that $\mathcal{T}$ is a topology and that $\{a\} \in \mathcal{T}$, $\{b\} \in \mathcal{T}$, and $\{c\} \in \mathcal{T}$.

We are required to prove that $\mathcal{T}$ is the discrete topology; that is, we are required to prove (by Definitions 1.1.6) that $\mathcal{T}$ contains all subsets of X. Remember that $\mathcal{T}$ is a topology and so satisfies conditions (i), (ii) and (iii) of Definitions 1.1.1.

So we shall begin our proof by writing down all of the subsets of X.

The set X has 3 elements and so it has 2^3 distinct subsets. They are: $S_1 = \emptyset$, $S_2 = \{a\}$, $S_3 = \{b\}$, $S_4 = \{c\}$, $S_5 = \{a, b\}$, $S_6 = \{a, c\}$, $S_7 = \{b, c\}$, and $S_8 = \{a, b, c\} = X$.

We are required to prove that each of these subsets is in $\mathcal{T}$. As $\mathcal{T}$ is a topology, Definitions 1.1.1 (i) implies that X and $\emptyset$ are in $\mathcal{T}$; that is, $S_1 \in \mathcal{T}$ and $S_8 \in \mathcal{T}$.

We are given that $\{a\} \in \mathcal{T}$, $\{b\} \in \mathcal{T}$ and $\{c\} \in \mathcal{T}$; that is, $S_2 \in \mathcal{T}$, $S_3 \in \mathcal{T}$ and $S_4 \in \mathcal{T}$.

To complete the proof we need to show that $S_5 \in \mathcal{T}$, $S_6 \in \mathcal{T}$, and $S_7 \in \mathcal{T}$. But $S_5 = \{a, b\} = \{a\} \cup \{b\}$. As we are given that $\{a\}$ and $\{b\}$ are in $\mathcal{T}$, Definitions 1.1.1 (ii) implies that their union is also in $\mathcal{T}$; that is, $S_5 = \{a, b\} \in \mathcal{T}$.

Similarly $S_6 = \{a, c\} = \{a\} \cup \{c\} \in \mathcal{T}$ and $S_7 = \{b, c\} = \{b\} \cup \{c\} \in \mathcal{T}$. $\square$

In the introductory comments on this chapter we observed that mathematics is not a spectator sport. You should be an active participant. Of course your participation includes doing some of the exercises. But more than this is expected of you. You have to think about the material presented to you.

One of your tasks is to look at the results that we prove and to ask pertinent questions. For example, we have just shown that if each of the singleton sets $\{a\}, \{b\}$ and $\{c\}$ is in $\mathcal{T}$ and $X = \{a, b, c\}$, then $\mathcal{T}$ is the discrete topology. You should ask if this is but one example of a more general phenomenon; that is, if $(X, \mathcal{T})$ is any topological space such that $\mathcal{T}$ contains every singleton set, is $\mathcal{T}$ necessarily the discrete topology? The answer is "yes", and this is proved in Proposition 1.1.9.

1.1.9 Proposition. If $(X, \mathcal{T})$ is a topological space such that, for every $x \in X$, the singleton set $\{x\}$ is in $\mathcal{T}$, then $\mathcal{T}$ is the discrete topology.

Proof.

This result is a generalization of Example 1.1.8. Thus you might expect that the proof would be similar. However, we cannot list all of the subsets of X as we did in Example 1.1.8 because X may be an infinite set. Nevertheless we must prove that every subset of X is in $\mathcal{T}$.

At this point you may be tempted to prove the result for some special cases, for example taking X to consist of 4, 5 or even 100 elements. But this approach is doomed to failure. Recall our opening comments in this chapter where we described a mathematical proof as a watertight argument. We cannot produce a watertight argument by considering a few special cases, or even a very large number of special cases. The watertight argument must cover all cases. So we must consider the general case of an arbitrary non-empty set X. Somehow we must prove that every subset of X is in $\mathcal{T}$.

Looking again at the proof of Example 1.1.8 we see that the key is that every subset of X is a union of singleton subsets of X and we already know that all of the singleton subsets are in $\mathcal{T}$. This is also true in the general case.

We begin the proof by recording the fact that every set is a union of its singleton subsets. Let S be any subset of X. Then

$$S = \bigcup_{x \in S} \{x\}.$$

Since we are given that each $\{x\}$ is in $\mathcal{T}$, Definitions 1.1.1 (ii) and the above equation imply that $S \in \mathcal{T}$. As S is an arbitrary subset of X, we have that $\mathcal{T}$ is the discrete topology. $\square$

That every set S is a union of its singleton subsets is a result which we shall use from time to time throughout the book in many different contexts. Note that it holds even when $S = \emptyset$ as then we form what is called an empty union and get $\emptyset$ as the result.